

APPLIED MATHEMATICS-1

Dec 2017

Mumbai University Dec 2017 (CBCS) solutions.

Q.1) Answer the following

(20 marks)

1a) Separate into real and imaginary parts of $\cos^{-1}\left(\frac{3i}{4}\right)$.

(3 marks)

Ans. Let $a + ib = \cos^{-1}\left(\frac{3i}{4}\right)$ (1)

$$\therefore \cos(a + ib) = \frac{3i}{4}$$

$$\therefore \cos(a)\cos(ib) - \sin(a)\sin(ib) = \frac{3i}{4}$$

$$\cos(a)\cosh(b) - i\sin(a)\sinh(b) = 0 + \frac{3i}{4} \quad \{\because \cos(ix) = \cosh(x), \sin(ix) = \sinh(x)\}$$

Comparing Real and Imaginary terms on both sides,

$$\cos(a)\cosh(b) = 0 \quad \dots(2) \quad \& \quad -\sin(a)\sinh(b) = \frac{3}{4} \quad \dots(3)$$

From (2), $\cos(a) = 0$ or $\cosh(b) = 0$,

$$\therefore a = \frac{\pi}{2} \quad \dots(4)$$

From (3) & (4), $-\sin\left(\frac{\pi}{2}\right)\sinh(b) = \frac{3}{4}$

$$\therefore 1 \cdot \sinh(b) = \frac{-3}{4}$$

$$\therefore b = \sinh^{-1}\left(\frac{-3}{4}\right)$$

$$= \log\left[\left(\frac{-3}{4}\right) + \sqrt{\left(\frac{-3}{4}\right)^2 + 1}\right] \quad \{\because \sinh^{-1}z = \log(z + \sqrt{z^2 + 1})\}$$

$$= \log\left[\left(\frac{-3}{4}\right) + \sqrt{\frac{9}{16} + 1}\right]$$

$$= \log\left[\left(\frac{-3}{4}\right) + \frac{5}{4}\right]$$

$$= \log\frac{1}{2}$$

$$= \log 2^{-1}$$

$$\therefore b = -\log 2 \quad \dots(5)$$

Substituting (4) & (5) in (1), $\cos^{-1}\left(\frac{3i}{4}\right) = \frac{\pi}{2} - i \log 2$

Comparing Real and Imaginary terms on both sides,

Real part = $a = \frac{\pi}{2}$

Imaginary part = $b = -\log 2$

1b) Show that the matrix A is unitary where $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$$

(3 marks)

Ans: $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} \alpha + i\gamma & \beta + i\delta \\ -\beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$\therefore A^\theta = \overline{A^T} = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

Given, A is Unitary

$$\therefore AA^\theta = 1$$

$$\therefore \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \times \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} (\alpha + i\gamma)(\alpha - i\gamma) + (-\beta + i\delta)(-\beta - i\delta) & (\alpha + i\gamma)(\beta - i\delta) + (-\beta + i\delta)(\alpha + i\gamma) \\ (\beta + i\delta)(\alpha - i\gamma) + (\alpha - i\gamma)(-\beta - i\delta) & (\beta + i\delta)(\beta - i\delta) + (\alpha - i\gamma)(\alpha + i\gamma) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \dots(1)$$

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Consider,

$$(\alpha + i)(\alpha - i) + (-\beta + i\delta)(-\beta - i\delta) = \alpha^2 + \beta^2 - i^2\delta^2$$

$$(\alpha + i)(\alpha - i) + (-\beta + i\delta)(-\beta - i\delta) = \alpha^2 + \beta^2 + \delta^2 \quad \dots(2)$$

$$\text{And, } (\alpha + i)(\beta - i\delta) + (-\beta + i\delta)(\alpha + i) = (\alpha\beta - i\alpha\delta + i\beta\gamma - i^2\gamma\delta) + (-\alpha\beta - i\beta\gamma + i\alpha\delta + i^2\gamma\delta) = 0 \quad \dots(3)$$

Similarly,

$$(\beta + i\delta)(\alpha - i) + (\alpha - i\gamma)(-\beta - i\delta) \quad \dots(4)$$

$$(\beta + i\delta)(\beta - i\delta) + (\alpha - i\gamma)(\alpha + i\gamma) = \beta^2 + \gamma^2 + \delta^2 \quad \dots(5)$$

Substituting (2),(3),(4)&(5) in (1),

$$\begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing corresponding terms, we get,

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

1c) If $z = \tan(y + ax) + (y - ax)^{3/2}$ then show that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$ (3 marks)

Ans: $z = \tan(y + ax) + (y - ax)^{3/2}$... (1)

Differentiate partially w.r.t.x, $\frac{\partial z}{\partial x} = \sec^2(y + ax) \cdot a + \frac{3}{2}(y - ax)^{1/2} \cdot (-a)$

$$\therefore \frac{\partial z}{\partial x} = a \sec^2(y + ax) - \frac{3a}{2}(y - ax)^{1/2}$$

Again, differentiate partially w.r.t.x,

$$\frac{\partial^2 z}{\partial x^2} = 2a^2 \sec^2(y + ax) \cdot \tan(y + ax) - \frac{3}{4}a^2 (y - ax)^{-1/2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = a^2 \left[2 \sec^2(y + ax) \tan(y + ax) - \frac{3}{4}(y - ax)^{-1/2} \right] \quad \dots(2)$$

Differentiate (1) partially w.r.t.y, $\frac{\partial z}{\partial y} = \sec^2(y + ax) \cdot 1 + \frac{3}{2}(y - ax)^{1/2}$

Again, differentiate partially w.r.t.y, $\frac{\partial^2 z}{\partial y^2} = 2 \sec^2(y + ax) \cdot \tan(y + ax) - \frac{3}{4}(y - ax)^{-1/2}$... (3)

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From (2)&(3),

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$$

1d) If $x=uv$, $y=\frac{u}{v}$. Prove that $JJ'=1$.

(3marks)

Ans: $x=uv$...(1)

$$\therefore x_u = \frac{\partial x}{\partial u} = v \quad \text{and} \quad x_v = \frac{\partial x}{\partial v} = u \quad \dots(2)$$

And, $y = \frac{u}{v}$...(3)

$$\therefore y_u = \frac{\partial y}{\partial u} = \frac{1}{v} \quad \text{and} \quad y_v = \frac{\partial y}{\partial v} = u \frac{-1}{v^2} \quad \dots(4)$$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= x_u y_v - x_v y_u$$

$$= v u \frac{-1}{v^2} - u \frac{1}{v} \quad \dots(\text{From 2 \& 4})$$

$$= \frac{-u}{v} - \frac{u}{v}$$

$$= \frac{-2u}{v}$$

$$\therefore J = -2y \quad \dots(5)$$

From (3), $u = vy$...(6)

Substituting 'u' in (1) we get, $x = (vy)v$

$$\frac{x}{y} = v^2$$

$$\therefore v = \frac{\sqrt{x}}{\sqrt{y}} = x^{1/2} y^{-1/2} \quad \dots(7)$$

$$\therefore v_x = y^{-1/2} \cdot \frac{1}{2} x^{-1/2} \quad \text{and} \quad v_y = x^{1/2} \cdot \frac{-1}{2} y^{-3/2} \quad \dots(8)$$

From (6) and (7), $u = (x^{1/2} y^{-1/2})y$

$$\therefore u = x^{1/2} y^{1/2}$$

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$$\therefore u_x = y^{1/2} \quad \text{and} \quad u_y = x^{1/2} \cdot \frac{1}{2} y^{-1/2} \quad \dots(9)$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= u_x v_y - u_y v_x$$

$$= \left(y^{1/2} \cdot \frac{1}{2} x^{-1/2} \right) \left(x^{1/2} \cdot \frac{-1}{2} y^{-3/2} \right) - \left(x^{1/2} \cdot \frac{1}{2} y^{-1/2} \right) \left(y^{-1/2} \cdot \frac{1}{2} x^{-1/2} \right) \quad \dots(\text{From 8 \& 9})$$

$$= \frac{-1}{4} x^{\frac{-1}{2} + \frac{1}{2}} \cdot y^{\frac{1}{2} - \frac{3}{2}} - \frac{-1}{4} x^{\frac{1}{2} - \frac{1}{2}} \cdot y^{\frac{-1}{2} - \frac{1}{2}}$$

$$= \frac{-1}{4} \cdot y^{-1} - \frac{1}{4} \cdot y^{-1}$$

$$= \frac{-2}{4} \cdot y^{-1}$$

$$\therefore J' = \frac{-1}{2y} \quad \dots(10)$$

From (5) and (10), $J \cdot J' = -2y \cdot \frac{-1}{2y}$

$$\therefore J \cdot J' = 1$$

1e) Find the nth derivative of $\frac{x^3}{(x+1)(x-2)}$.

(4 marks)

Ans: Let $y = \frac{x^3}{(x+1)(x-2)} = \frac{x^3}{x^2 - x - 2}$

Consider, $x^2 - x - 2 \overline{) x^3 + 0x^2 + 0x + 0}$

$$\underline{x^3 - x^2 - 2x}$$

$$x^2 + 2x + 0$$

$$\underline{x^2 - x - 2}$$

$$3x + 2$$

$$\therefore y = x + 1 + \frac{3x+2}{x^2-x-2}$$

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$$\therefore y = x + 1 + \frac{3x+2}{(x+1)(x-2)}$$

$$\therefore y = x + 1 + \frac{1/3}{(x+1)} + \frac{8/3}{(x-2)} \quad (\text{By Partial Fraction})$$

Taking n^{th} order derivative, $y_n = 0+0+\frac{1}{3} \cdot \frac{n! \cdot 1^n (-1)^n}{(x+1)^{n+1}} + \frac{8}{3} \cdot \frac{n! \cdot 1^n (-1)^n}{(x-2)^{n+1}}$

$$\left\{ \text{If } y = \frac{1}{ax+b} \text{ then } y_n = \frac{n! a^n (-1)^n}{(ax+b)^{n+1}} \right\}$$

$$\therefore y_n = \frac{n! (-1)^n}{3} \left[\frac{1}{(x+1)^{n+1}} + \frac{8}{(x-2)^{n+1}} \right]$$

1f) Using the matrix $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ decode the message of matrix $C = \begin{bmatrix} 4 & 11 & 12 & -2 \\ -4 & 4 & 9 & -2 \end{bmatrix}$

(4 marks)

Ans: Encoding Matrix $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$... (1)

Given, $C = \begin{bmatrix} 4 & 11 & 12 & -2 \\ -4 & 4 & 9 & -2 \end{bmatrix}$

Step 1:

Writing the numbers in C matrix column wise gives the encoded message.

\therefore Encoded Message = 4 -4 11 4 12 9 -2 -2

This Encoded message is transmitted.

Assume there is no corruption of data, the message at the receiving end is 4 -4 11 4 12 9 -2 -2

This message is decoded

Step 2:

We know, if $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $P^{-1} = \frac{1}{|P|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

From (1), $|A| = -1 + 2 = 1$... (2)

\therefore Decoding matrix $A^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ (From 2) ... (3)

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$$\text{From (2) \& (3), } A^{-1}C = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 11 & 12 & -2 \\ -4 & 4 & 9 & -2 \end{bmatrix}$$

$$\therefore A^{-1}C = \begin{bmatrix} 4 + 8 & 11 - 8 & 12 - 18 & -2 + 4 \\ 4 + 4 & 11 - 4 & 12 - 9 & -2 + 2 \end{bmatrix}$$

$$\therefore A^{-1}C = \begin{bmatrix} 12 & 3 & 6 & 2 \\ 8 & 7 & 3 & 0 \end{bmatrix}$$

Step 3:

Considering the numbers column-wise we get,

12 8 3 7 -6 3 2 0

$$\text{Decoded Message} = 12\ 8\ 3\ 7\ -6\ 3\ 2\ 0 \text{ or } \begin{bmatrix} 12 & 3 & 6 & 2 \\ 8 & 7 & 3 & 0 \end{bmatrix}$$

Q.2)

(20 marks)

2a) If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$ then find a,b,c,d.

(6 marks)

$$\text{Ans: We know, } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \text{ and } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \dots(1)$$

$$\text{Consider, } \sin^4\theta\cos^3\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \times \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (\text{From 1})$$

$$= \frac{1}{2^4 i^4 2^3} \times (e^{i\theta} - e^{-i\theta})(e^{i\theta} - e^{-i\theta})^3 (e^{i\theta} + e^{-i\theta})^3$$

$$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta}) [(e^{i\theta})^2 - (e^{-i\theta})^2]^3$$

$$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta}) [(e^{2i\theta}) - (e^{-2i\theta})]^3$$

$$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta}) [(e^{2i\theta})^3 - 3(e^{2i\theta})^2(e^{-2i\theta}) + 3(e^{2i\theta})(e^{-2i\theta})^2 - (e^{-2i\theta})^3]$$

$$= \frac{1}{2^7} \times (e^{i\theta} - e^{-i\theta}) [e^{6i\theta} - 3e^{2i\theta} + 3e^{-2i\theta} - e^{-6i\theta}]$$

$$= \frac{1}{2^7} \times [e^{7i\theta} - 3e^{3i\theta} + 3e^{-i\theta} - e^{-5i\theta} - e^{5i\theta} + 3e^{\theta} - 3e^{-3i\theta} + e^{-7i\theta}]$$

$$= \frac{1}{2^7} [(e^{7i\theta} + e^{-7i\theta}) - (e^{5i\theta} + e^{-5i\theta}) - 3(e^{3i\theta} + e^{-3i\theta}) + 3(e^{i\theta} + e^{-i\theta})]$$

$$= \frac{1}{128} \times 2\cos 7\theta - \frac{1}{128} \times 2\cos 5\theta - \frac{1}{128} \times 6\cos 3\theta + \frac{1}{128} \times 6\cos\theta$$

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$$\therefore \sin^4\theta \cos^3\theta = \frac{3}{64}\cos\theta - \frac{3}{64}\cos 3\theta - \frac{1}{64}\cos 5\theta + \frac{1}{64}\cos 7\theta \quad \dots(2)$$

$$\text{But , given, } \sin^4\theta \cos^3\theta = a\cos\theta + b\cos 3\theta + c\cos 5\theta + d\cos 7\theta \quad \dots(3)$$

Comparing (2) & (3),
$$a = \frac{3}{64} ; b = \frac{-3}{64} ; c = \frac{-1}{64} ; d = \frac{1}{64} ;$$

2b) Using Newton Raphson method solve $3x - \cos x - 1 = 0$. Correct upto 3 decimal places.

(6 marks)

Ans: Let $f(x) = 3x - \cos x - 1$

$$\therefore f'(x) = 3 + \sin x - 0$$

When $x = 0, f(0) = 3(0) - \cos 0 - 1 = -2$

When $x = 1, f(1) = 3(1) - \cos 1 - 1 = 1.4597$

\therefore Roots of $f(x)$ lies between 0 and 1.

Let initial value $x_0 = 0$

By Newton-Raphson's Method $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n(3 + \sin x_n) - (3x_n - \cos x_n - 1)}{3 + \sin x_n}$$

$$= \frac{3x_n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3 + \sin x_n}$$

$$\therefore x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad \dots(1)$$

Iteration 1: Put $n = 0$ in (1)

$$\therefore x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0 + \cos 0 + 1}{3 + \sin 0} = 0.6667$$

Iteration 2: Put $n = 1$ in (1)

$$\therefore x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6667 \sin(0.6667) + \cos(0.6667) + 1}{3 + \sin(0.6667)} = 0.6075$$

Iteration 3: Put $n = 2$ in (1)

$$\therefore x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2} = \frac{0.6075 \sin(0.6075) + \cos(0.6075) + 1}{3 + \sin(0.6075)} = 0.6071$$

Iteration 4: Put $n = 3$ in (1)

$$\therefore x_4 = \frac{x_3 \sin x_3 + \cos x_3 + 1}{3 + \sin x_3} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} = 0.6071$$

Hence, Root of $3x - \cos x - 1 = 0$ is 0.6071

2c) Find the stationary points of the function $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ & also find maximum and minimum values of the function. (8 marks)

Ans: Let $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$...(1)

$$\therefore f_x = 3x^2 + 3y^2 - 6x - 0 + 0$$

$$\therefore r = f_{xx} = 6x - 6$$
 ...(2)

Also, $f_y = 0 + 6xy - 0 - 6y + 0$

$$\therefore t = f_{yy} = 6x - 6$$
 ...(3)

$$\therefore s = f_{xy} = 0 + 6y - 0$$
 ...(4)

Put $f_x = 0$ and $f_y = 0$

$$\therefore 3x^2 + 3y^2 - 6x = 0$$

$$\therefore x^2 + y^2 - 2x = 0$$
 ...(5)

And, $6xy - 6y = 0$

$$\therefore 6y(x-1) = 0$$

$$\therefore y = 0 \text{ or } x = 1$$

Case I : When $x = 1$

From (5), $1^2 + y^2 - 2(1) = 0$

$$\therefore y^2 - 1 = 0$$

$$\therefore y = \pm 1$$

Case II: When $y = 0$

From (5), $x^2 + 0 - 2x = 0$

$$\therefore x(x-2) = 0$$

$$\therefore x = 0 \text{ or } x = 2$$

\therefore Stationary points are $(1,1);(1,-1);(0,0);(2,0)$:

(i)At $(1,1)$

From (2), $r = 6(1) - 6 = 0$

$\therefore f$ is neither maximum or minimum at $(1,1)$

(ii)At $(1,-1)$

From (2), $r = 6(1) - 6 = 0$

$\therefore f$ is neither maximum or minimum at $(1,-1)$

(iii)At $(0,0)$

From (2), $r = 6(0) - 6 = -6 < 0$

From (3), $t = 6(0) - 6 = -6$

From (4), $s = 6(0) = 0$

$$\therefore rt - s^2 = (-6)(-6) - 0 = 36 > 0$$

$\therefore f$ has maximum at $(0,0)$

From (1), Maximum value of f

$$\therefore f = (0)^3 + 3(0)(0)^2 - 3(0)^2 - 3(0)^2 + 4 = 4$$

(iv) At (2,0)

From (2), $r = 6(2) - 6 = 6 < 0$

From (3), $t = 6(2) - 6 = 6$

From (4), $s = 6(0) = 0$

$\therefore rt - s^2 = (6)(6) - 0 = 36 > 0$

$\therefore f$ has maximum at (2,0)

From (1), Minimum value of f

$\therefore f = (2)^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4 = 0$

Hence the function has

Maximum at (0,0) and Maximum value = 4
 Minimum at (2,0) and Minimum value = 0

Q.3)

(20 marks)

3a) Show that $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$

(6 marks)

Ans: LHS = $x \operatorname{cosec} x$

$$= \frac{x}{\sin x}$$

$$= \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

$$= \frac{x}{x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right)}$$

$$= \left[\left(1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \dots \right) \right) \right]^{-1}$$

$$= 1 + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \dots \right) + \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \dots \right)^2 + \dots \quad \{ \because (1 - y)^{-1} = 1 + y + y^2 + y^3 + \dots \}$$

$$= 1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!} \right)^2 + \dots$$

$$= 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$$

$$\therefore x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$$

3b) Reduce matrix to PAQ normal form and find 2 non-Singular matrices P & Q.

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

(6 marks)

Ans: $A_{3 \times 4} = I_{3 \times 3} \times A_{3 \times 4} \times I_{4 \times 4}$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_1; R_3 - R_1; \quad \rightarrow \quad \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - 2C_1; C_3 + C_1; C_4 - 2C_1; \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_4 + C_2; \frac{1}{2} C_3 \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

LHS is the required PAQ form .

$$\text{Here, } P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -2 & 1/2 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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3c) If $y = \cos(m \sin^{-1} x)$. Prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$.

(8 marks)

Ans: $y = \cos(m \sin^{-1} x)$...(1)

Differentiating w.r.t. 'x', $y_1 = -\sin(m \sin^{-1} x) \cdot m \cdot \frac{1}{\sqrt{1-x^2}}$

$$\therefore \sqrt{1-x^2} y_1 = -m \sin(m \sin^{-1} x)$$

On Squaring, $(1-x^2)y_1^2 = m^2 \sin^2(m \sin^{-1} x)$

$$\therefore (1-x^2)y_1^2 = m^2 [1 - \cos^2(m \sin^{-1} x)]$$

$$\therefore (1-x^2)y_1^2 = m^2 [1 - y^2] \quad \text{(From 1)}$$

Again differentiating w.r.t. 'x', $(1-x^2)2y_1y_2 + y_1^2(-2x) = m^2(0 - 2yy_1)$

$$\therefore (1-x^2)y_2 - xy_1 = -m^2y \quad \text{(Dividing by } 2y_1)$$

Applying Leibnitz theorem, $\{y_n = u_n v + n u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + {}^n C_3 u_{n-3} v_3 + \dots\}$

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!} \right] - [xy_{n+1} + ny_n] = -m^2y_n$$

$$\therefore (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!} - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - xy_n(2n+1) + (-n^2 + n - n + m^2)y_n = 0$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

Q.4)

(20 marks)

4a) State and Prove Euler's Theorem for three variables.

(6 marks)

Ans: Euler's theorem:

Statement: If 'u' is a homogenous function of three variables x, y, z of degree 'n' then Euler's theorem

States that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$$

Proof:

Let $u = f(x, y, z)$ be the homogenous function of degree 'n'.

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Let $X = xt, Y = yt, Z = zt$

$$\therefore \frac{\partial X}{\partial t} = x, \frac{\partial Y}{\partial t} = y, \frac{\partial Z}{\partial t} = z \quad \dots(1)$$

At $t = 1,$...(2)

$X = x, Y = y, Z = z$

$$\therefore \frac{\partial f}{\partial X} = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial y}, \frac{\partial f}{\partial Z} = \frac{\partial f}{\partial z} \quad \dots(3)$$

Now, $f(X, Y, Z) = t^n f(x, y, z)$...(4)

$$\therefore f \rightarrow X, Y, Z \rightarrow x, y, z, t$$

Differentiating (4) partially w.r.t. 't', $\frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial t} + \frac{\partial f}{\partial Z} \cdot \frac{\partial Z}{\partial t} = nt^{n-1} f(x, y, z)$

$$\therefore \frac{\partial f}{\partial X} \cdot x + \frac{\partial f}{\partial Y} \cdot y + \frac{\partial f}{\partial Z} \cdot z = (nt)^{n-1} f(x, y, z) \quad \text{(From 1,2 & 3)}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$$

4b) Show that all roots of $(x + 1)^6 + (x - 1)^6 = 0$ are given by $-icot \frac{(2k+1)\pi}{12}$ where $k=0,1,2,3,4,5$.

(6 marks)

Ans: $(x + 1)^6 + (x - 1)^6 = 0$

$$\therefore (x + 1)^6 = -(x - 1)^6$$

$$\therefore \frac{(x+1)^6}{(x-1)^6} = -1$$

$$\therefore \left(\frac{x+1}{x-1}\right)^6 = e^{i\pi} \quad \left\{ \because e^{i\pi} = \cos\pi + i\sin\pi = -1(\theta) \neq -1 \right\} \text{ (Principal value)}$$

$$\therefore \left(\frac{x+1}{x-1}\right)^6 = e^{i(\pi+2k\pi)}, \quad k = 0, 1, 2, 3, 4, 5 \quad \text{(General Value)}$$

$$\therefore \frac{x+1}{x-1} = e^{i\pi(1+2k)/6} \quad \dots(1)$$

Let $2\theta = \frac{\pi(1+2k)}{6}$...(2)

$$\therefore \text{From (1) \& (2), } \frac{x+1}{x-1} = e^{i2\theta}$$

$$\therefore \text{By Componendo - Dividendo, } \frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i2\theta}+1}{e^{i2\theta}-1}$$

$$\therefore \frac{2x}{2} = \frac{e^{i\theta}[e^{i\theta}+e^{-i\theta}]}{e^{i\theta}[e^{i\theta}-e^{-i\theta}]}$$

$$\therefore x = \frac{2\cos\theta}{2i\sin\theta} \quad \left\{ \because \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \text{ and } \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \right\}$$

$$\therefore x = \frac{1}{i} \cot\theta$$

$$\therefore x = -i\cot\frac{(2k+1)\pi}{12} \quad (\text{From 2) where } k = 0, 1, 2, 3, 4, 5$$

4c) Show that the following equations: $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ have no solutions unless $a + b + c = 0$ in which case they have infinitely many solutions. Find these solutions when $a=1$, $b=1$, $c=-2$. (8 marks)

Ans: Part I:

$$-2x + y + z = a$$

$$x - 2y + z = b$$

$$x + y - 2z = c$$

Writing the equations in the matrix form,

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$R_3+(R_1+R_2) \quad \rightarrow \quad \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ a + b + c \end{bmatrix} \quad \dots(1)$$

$$\text{Augmented matrix } [A|B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 0 & 0 & 0 & a + b + c \end{array} \right]$$

Number of unknowns = $n = 3$

Rank of A (r_A) = Number of non-zero rows in A = 2

Case I: No Solution

For which, $r_A < r_{AB}$

This is only possible, when 'a+b+c≠0' upon which,

$$\text{Rank of } [A|B] = (r_{AB}) = 3$$

Case II: Infinite Solution

For which, $r_A = r_{AB} < n$ (i.e. < 3)

This is only possible, when 'a+b+c=0' upon which,

$$\text{Rank of } [A|B] = r_{AB} = 2$$

Part II: Put $a = 1, b = 1, c = -2$, in (1)

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$R_2 - R_1; \quad \rightarrow \quad \begin{bmatrix} -2 & 1 & 1 \\ 3 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \dots(2)$$

Here, $n - r_A = 3 - 2 = 1$

We have to assume one unknown.

Let $y = t$ ($\neq 0$)

On expanding (2), $3x - 3y = 0$

$$\therefore x - y = 0$$

$$\therefore x = y = t$$

And, $-2x + y + z = 1$

$$\therefore -2t + t + z = 1$$

$$\therefore z = 1 + t$$

Hence, the solution is

$$x = t, y = t, z = 1 + t \text{ (Infinite Solution)}$$

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Q5)

5a) If $z = f(x, y)$. $x = r \cos \theta$, $y = r \sin \theta$. prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$ (6 marks)

Ans: $x = r \cos \theta$ and $y = r \sin \theta$... (1)

Differentiating partially w.r.t. ' θ '. $\frac{\partial x}{\partial \theta} = -r \sin \theta$; $\frac{\partial y}{\partial \theta} = r \cos \theta$; ... (2)

Differentiating partially w.r.t. ' r ', $\frac{\partial x}{\partial r} = \cos \theta$; $\frac{\partial y}{\partial r} = \sin \theta$... (3)

Now, $z \rightarrow x, y \rightarrow r, \theta$

By Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$

$\therefore \frac{\partial z}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y}$ (From 3) ... (4)

Similarly, By Chain Rule, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$

$\therefore \frac{\partial z}{\partial \theta} = -r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}$ (From 2) ... (5)

RHS = $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$

= $\left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2} \left(-r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}\right)^2$ (From 4 & 5)

= $\cos^2 \theta \left(\frac{\partial z}{\partial x}\right)^2 + 2 \sin \theta \frac{\partial z}{\partial x} \cdot \cos \theta \frac{\partial z}{\partial y} + \sin^2 \theta \left(\frac{\partial z}{\partial y}\right)^2 + \sin^2 \theta \left(\frac{\partial z}{\partial x}\right)^2 - 2 \sin \theta \frac{\partial z}{\partial x} \cos \theta \frac{\partial z}{\partial y} + \cos^2 \theta \left(\frac{\partial z}{\partial y}\right)^2$

= $\left(\frac{\partial z}{\partial x}\right)^2 (\cos^2 \theta + \sin^2 \theta) + \left(\frac{\partial z}{\partial y}\right)^2 (\cos^2 \theta + \sin^2 \theta)$

= $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

= LHS

$$\text{Hence, } \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

5b) If $\cosh x = \sec \theta$ prove that (i) $x = \log(\sec \theta + \tan \theta)$. (ii) $\theta = \frac{\pi}{2} \tan^{-1}(e^{-x})$ (6 marks)

Ans: (i) $\cosh x = \sec \theta$

$$\therefore \frac{e^x + e^{-x}}{2} = \sec \theta \quad \left\{ \because \cosh x = \frac{e^x + e^{-x}}{2} \right\}$$

$$\therefore e^x + \frac{1}{e^x} = 2 \sec \theta$$

$$\therefore (e^x)^2 + 1 = 2 \sec \theta e^x$$

$$\therefore (e^x)^2 - 2 \sec \theta e^x + 1 = 0$$

$$\therefore e^x = \frac{2 \sec \theta \pm \sqrt{4 \sec^2 \theta - 4}}{2} \quad \left\{ \because \text{Using, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right\}$$

$$\therefore e^x = \frac{2 \sec \theta \pm \sqrt{4(\sec^2 \theta - 1)}}{2}$$

$$\therefore e^x = \frac{2 \sec \theta \pm 2 \tan \theta}{2}$$

$$\therefore e^x = \sec \theta \pm \tan \theta$$

Considering only positive root,

$$\therefore e^x = \sec \theta + \tan \theta \quad \dots(1)$$

$$\therefore x = \log(\sec \theta + \tan \theta)$$

(ii) From (1), $e^x = \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}$

$$\therefore e^x = \frac{1 + \sin \theta}{\cos \theta}$$

$$\therefore \frac{1}{e^x} = \frac{\cos \theta}{1 + \sin \theta}$$

$$\therefore e^{-x} = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}$$

Put $\alpha = \frac{\pi}{2} - \theta$...(2)

$$\therefore e^{-x} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\therefore e^{-x} = \frac{2\sin(\alpha/2)\cos(\alpha/2)}{2\cos^2(\alpha/2)} \quad \{\because 2\sin A \cos A = \sin 2A; 1 + \cos 2A = 2\cos^2 A\}$$

$$\therefore e^x = \tan\left(\frac{\alpha}{2}\right)$$

$$\therefore \tan^{-1}(e^{-1}) = \frac{\alpha}{2}$$

$$\therefore 2\tan^{-1}(e^{-1}) = \alpha$$

$$\therefore 2\tan^{-1}(e^{-1}) = \frac{\pi}{2} - \theta \quad \text{(From 2)}$$

$$\therefore \theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-1})$$

5c) Solve by Gauss Jacobi Iteration Method: $5x - y + z = 10$, $2x + 4y = 12$, $x + y + 5z = -1$.

(8 marks)

Ans: From 1st equation, $5x = 10 + y - z$

$$\therefore x = \frac{1}{5}(10 + y - z) = 0.2(10 + y - z)$$

Similarly,

From 2nd equation, $x + 2y = 6$

$$\therefore 2y = 6 - x$$

$$y = \frac{1}{2}(6 - x) = 0.5(6 - x) \text{ and,}$$

$$z = 0.2(-1 - x - y) = -0.2(1 + x + y)$$

Iteration 1:

Put $x_0 = y_0 = z_0$

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$$\therefore x_1 = 0.2(10 + y_0 - z_0) = 0.2(10 + 0 - 0) = 2$$

$$\therefore y_1 = 0.5(6 - x_0) = 0.5(6 - 0) = 3$$

$$\therefore z_1 = -0.2(1 + x_0 + y_0) = -0.2(1 + 0 + 0) = -0.2$$

Iteration 2:

$$\text{Put } x_1 = 2; y_1 = 3; z_1 = -0.2$$

$$\therefore x_2 = 0.2(10 + y_1 - z_1) = 0.2(10 + 3 + 0.2) = 2.64$$

$$\therefore y_2 = 0.5(6 - x_1) = 0.5(6 - 2) = 2$$

$$\therefore z_2 = -0.2(1 + x_1 + y_1) = -0.2(1 + 2 + 3) = -1.2$$

Iteration 3:

$$\text{Put } x_2 = 2.64; y_2 = 2; z_2 = -1.2$$

$$\therefore x_3 = 0.2(10 + y_2 - z_2) = 0.2(10 + 2 + 1.2) = 2.64$$

$$\therefore y_3 = 0.5(6 - x_2) = 0.5(6 - 2.64) = 1.68$$

$$\therefore z_3 = -0.2(1 + x_2 + y_2) = -0.2(1 + 2.64 + 2) = -1.128$$

Iteration 4:

$$\text{Put } x_3 = 2.64; y_3 = 1.68; z_3 = -1.128$$

$$\therefore x_4 = 0.2(10 + y_3 - z_3) = 0.2(10 + 1.68 + 1.128) = 2.5615$$

$$\therefore y_4 = 0.5(6 - x_3) = 0.5(6 - 2.64) = 1.68$$

$$\therefore z_4 = -0.2(1 + x_3 + y_3) = -0.2(1 + 2.64 + 1.68) = -1.0640$$

Iteration 5:

$$\text{Put } x_4 = 2.5616; y_4 = 1.68; z_4 = -1.0640$$

$$\therefore x_5 = 0.2(10 + y_4 - z_4) = 0.2(10 + 1.68 + 1.0640) = 2.5488$$

$$\therefore y_5 = 0.5(6 - x_4) = 0.5(6 - 2.5616) = 1.7172$$

$$\therefore z_5 = -0.2(1 + x_4 + y_4) = -0.2(1 + 2.5616 + 1.68) = -1.0483$$

Iteration 6:

Put $x_5 = 2.5488$; $y_5 = 1.7192$; $z_5 = -1.0483$

$$\therefore x_6 = 0.2(10 + y_5 - z_5) = 0.2(10 + 1.7192 + 1.0483) = 2.5535$$

$$\therefore y_6 = 0.5(6 - x_5) = 0.5(6 - 2.5488) = 1.7256$$

$$\therefore z_6 = -0.2(1 + x_5 + y_5) = -0.2(1 + 2.5488 + 1.7192) = -1.0536$$

Iteration 7:

Put $x_6 = 2.5535$; $y_6 = 1.7256$; $z_6 = -1.0536$

$$\therefore x_7 = 0.2(10 + y_6 - z_6) = 0.2(10 + 1.7256 - 1.0536) = 2.5558$$

$$\therefore y_7 = 0.5(6 - x_6) = 0.5(6 - 2.5535) = 1.7232$$

$$\therefore z_7 = -0.2(1 + x_6 + y_6) = -0.2(1 + 2.5535 + 1.7256) = -1.0558$$

Iteration 8:

Put $x_7 = 2.5558$; $y_7 = 1.7232$; $z_7 = -1.0558$

$$\therefore x_8 = 0.2(10 + y_7 - z_7) = 0.2(10 + 1.7232 - 1.0558) = 2.5558$$

$$\therefore y_8 = 0.5(6 - x_7) = 0.5(6 - 2.5558) = 1.7221$$

$$\therefore z_8 = -0.2(1 + x_7 + y_7) = -0.2(1 + 2.5558 + 1.7232) = -1.0558$$

Hence, by Gauss Jacobi Iteration Method, the solution is

$x = 2.5558, y = 1.7221, z = -1.0558$

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Q.6)

(20 marks)

6a) Prove that $\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)$

(6 marks)

Ans: We know, $\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$

$$= \frac{(e^\theta - e^{-\theta})/2}{(e^\theta + e^{-\theta})/2}$$

$$\therefore \tanh\theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}$$

Put $\theta = \log x$, ...(1)

$$\therefore \tanh(\log x) = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}$$

$$= \frac{e^{\log x} - e^{\log x^{-1}}}{e^{\log x} + e^{\log x^{-1}}}$$

$$= \frac{x - x^{-1}}{x + x^{-1}}$$

$$= \frac{x(1 - x^{-2})}{x(1 + x^{-2})} \quad \dots(2)$$

Let $y = \cos^{-1}[\tanh(\log x)]$

$$= \cos^{-1}\left[\frac{1 - x^{-2}}{1 + x^{-2}}\right] \quad \text{(From 2)}$$

$$= \cos^{-1}\left[\frac{1 - (x^{-1})^{-2}}{1 + (x^{-1})^{-2}}\right]$$

Put $x^{-1} = \tan\theta$

$$\therefore y = \cos^{-1}\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2 \tan^{-1}\left(\frac{1}{x}\right) \quad \text{(From 1)}$$

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$$= 2\cot^{-1}x$$

$$= 2\left(\frac{\pi}{2} - \tan^{-1}x\right)$$

$$= \pi - 2\tan^{-1}x$$

$$= \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)$$

$$\text{Hence, } \boxed{\cos^{-1}[\tanh(\log x)] = \pi - 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)}$$

6b) If $y = e^{2x} \sin \frac{x}{2} \cos \frac{x}{2} \sin 3x$. Find y_n .

(6 marks)

$$\text{Ans: } y = e^{2x} \sin \frac{x}{2} \cos \frac{x}{2} \sin 3x \times \frac{2}{2}$$

$$= \frac{1}{2} e^{2x} \left[\sin 2\left(\frac{x}{2}\right) \right] \sin 3x \quad \{\because 2\sin A \cos A = \sin 2A\}$$

$$= \frac{1}{2} e^{2x} \sin x \sin 3x \times \frac{2}{2}$$

$$= \frac{1}{4} e^{2x} [\cos(3x - x) - \cos(3x + x)] \quad \{\because 2\sin A \sin B = \cos(A - B) - \cos(A + B)\}$$

$$\therefore y = \frac{1}{4} [e^{2x} \cos 2x - e^{2x} \cos 4x]$$

$$\text{Taking } n^{\text{th}} \text{ order derivative, } y_n = \frac{1}{4} \left\{ \frac{d^n}{dx^n} (e^{2x} \cos 2x) - \frac{d^n}{dx^n} (e^{2x} \cos 4x) \right\} \quad \dots(1)$$

$$\text{We know, if } y = e^{ax} \cos(bx + c) \quad y_n = r^n e^{ax} \cos(bx + c + n\phi) \quad \dots(2)$$

Here $a = 2$, $c = 0$, $b_1 = 2$ and $b_2 = 4$

$$\therefore r_1 = \sqrt{a^2 + b_1^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \quad \text{and} \quad r_2 = \sqrt{a^2 + b_2^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \quad \dots(3)$$

$$\text{And, } \phi_1 = \tan^{-1} \frac{b_1}{a} = \tan^{-1} \frac{2}{2} = \tan^{-1} 1 = \frac{\pi}{4} \quad \& \quad \phi_2 = \tan^{-1} \frac{b_2}{a} = \tan^{-1} \frac{4}{2} = \tan^{-1} 2 \quad \dots(4)$$

\therefore From (1), (2), (3) and (4),

$$y_n = \frac{1}{4} \left\{ (2\sqrt{2})^n e^{2x} \cos(2x + 0 + n\phi_1) + (2\sqrt{5})^n e^{2x} \cos(4x + 0 + n\phi_2) \right\}$$

$$\therefore y = \frac{1}{4} e^{2x} \left[8^{n/2} \cos\left(2x + \frac{n\pi}{4}\right) + 20^{n/2} \cos(4x + n\phi) \right], \quad \text{where } \phi_2 = \tan^{-1} \frac{1}{2}$$

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6c) (i) Evaluate $\lim_{x \rightarrow 0} (\cot x)^{\sin x}$.

(4 marks)

(ii) Prove that $\log \left[\frac{\sin(x+iy)}{\sin(x-iy)} \right] = 2i \tan^{-1}(\cot x \tanh y)$.

(4 marks)

Ans: (i) Let $L = \lim_{x \rightarrow 0} (\cot x)^{\sin x}$

$$\therefore \log L = \log \left\{ \lim_{x \rightarrow 0} (\cot x)^{\sin x} \right\}$$

$$= \lim_{x \rightarrow 0} \{ \log (\cot x)^{\sin x} \}$$

$$= \lim_{x \rightarrow 0} \sin x \cdot \log (\cot x)$$

$$= \lim_{x \rightarrow 0} \frac{\log(\cot x)}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\cot x} - \operatorname{cosec}^2 x}{-\operatorname{cosec} x \cot x} \quad (\text{L' Hospital's Rule})$$

$$= \lim_{x \rightarrow 0} \tan x \cdot \frac{1}{\sin x} \cdot \tan x$$

$$= \lim_{x \rightarrow 0} \tan x \cdot \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x}$$

$$= \tan 0 \times \frac{1}{\cos 0}$$

$$\therefore \log L = 0$$

$$\therefore L = e^0$$

$$\therefore \lim_{x \rightarrow 0} (\cot x)^{\sin x} = 1$$

(ii) Consider, $\log[\sin(x+iy)] = \log[\sin x \cosh y + i \cos x \sinh y]$

$$\therefore \log[\sin(x+iy)] = \log[\sin x \cosh y + i \cos x \sinh y] \quad \{ \because \cos(ix) = \cosh x; \sin(ix) = i \sinh x; \}$$

$$\therefore \log[\sin(x+iy)] = \frac{1}{2} \log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] + i \tan^{-1} \left| \frac{\cos x \sinh y}{\sin x \cosh y} \right|$$

$$\left\{ \because \log(x + iy) = \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \left| \frac{y}{x} \right| \right\}$$

$$\therefore \log[\sin(x+iy)] = \frac{1}{2} \log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] + i \tan^{-1} |\cot x \tanh y|$$

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Taking Conjugate, $\log[\sin(x - iy)] = \frac{1}{2} \log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] - i \tan^{-1} \cot x \tanh y$

Now, $\log \left[\frac{\sin(x+iy)}{\sin(x-iy)} \right] = \log [\sin(x + iy)] - \log [\sin(x - iy)]$

$$= \left\{ \frac{1}{2} \log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] + i \tan^{-1} \cot x \tanh y \right\} -$$

$$\left\{ \frac{1}{2} \log[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y] - i \tan^{-1} \cot x \tanh y \right\}$$

$$= 2i \tan^{-1}(\cot x \tanh y)$$

$$\therefore \log \left[\frac{\sin(x+iy)}{\sin(x-iy)} \right] = 2i \tan^{-1}(\cot x \tanh y)$$