

**Q.1(a) Prove that  $\tanh^{-1}(\sin \theta) = \cosh^{-1}(\sec \theta)$  [3]**

**Ans :**        L.H.S    =  $\tanh^{-1}(\sin \theta)$

We know that ,  $\tanh^{-1}(x) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

$\therefore$  L.H.S    =  $\frac{1}{2} \log\left(\frac{1+\sin \theta}{1-\sin \theta}\right)$

R.H.S    =  $\cosh^{-1}(\sec \theta)$

We know that ,  $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$

$\therefore$  R.H.S    =  $\log(\sec \theta + \sqrt{\sec^2 \theta - 1})$

$= \log\left(\frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}\right)$         ..... $\{\sqrt{\sec^2 \theta - 1} = \tan \theta \frac{\sin \theta}{\cos \theta}\}$

$= \log\left(\frac{1+\sin \theta}{\cos \theta}\right)$

$= \log\left(\frac{1+\sin \theta}{\sqrt{1-\sin^2 \theta}}\right)$

$= \log\left(\frac{\sqrt{1+\sin \theta}}{\sqrt{1-\sin \theta}}\right)$

$= \frac{1}{2} \log\left(\frac{1+\sin \theta}{1-\sin \theta}\right)$

$\therefore \tanh^{-1}(\sin \theta) = \cosh^{-1}(\sec \theta)$

**Hence Proved .**

**(b) Prove that the matrix  $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$  is unitary. [3]**

**Ans: Let**     $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

**The matrix is unitary when  $A.A^{\theta} = I$  .**

$$\therefore A^\theta = \overline{(A^t)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & - \\ 1 + i & -1 & \end{bmatrix}^t = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & + \\ 1 & - & -1 \end{bmatrix}^i$$

$$\begin{aligned} \therefore \mathbf{A} \cdot \mathbf{A}^\theta &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & + \\ 1 & - & -1 \end{bmatrix}^i \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & + \\ 1 & - & -1 \end{bmatrix}^i \\ &= \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\boxed{\therefore \mathbf{A} \cdot \mathbf{A}^\theta = \mathbf{I}}$$

The given matrix is unitary is proved.

(c) If  $x=uv$  &  $y=\frac{u}{v}$  prove that  $JJ^1 = 1$  [3]

Ans:  $x=uv$  and  $y=\frac{u}{v}$

$\therefore x$  and  $y$  are function of  $u$  and  $v$ .

$$\therefore u = \sqrt{xy} \quad \therefore v = \sqrt{\frac{x}{y}} \quad \text{..... { from given eqns } }$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v} \quad \text{.....(1)}$$

$$J^1 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix} = \frac{-\sqrt{x}}{2\sqrt{xy}} = \frac{-v}{2u} \quad \text{.....(2)}$$

$$\therefore JJ^1 = \frac{-2u}{v} \times \frac{-v}{2u} = 1$$

$$\boxed{\therefore JJ^1 = 1}$$

Hence Proved.

(d) If  $z = \tan^{-1}\left(\frac{x}{y}\right)$ , where  $x=2t$ ,  $y=1-t^2$ , prove that  $\frac{dz}{dt} = \frac{2}{1+t^2}$  . [3]

**Ans:**  $z = \tan^{-1}\left(\frac{x}{y}\right)$        $x=2t$  and  $y=1-t^2$

∴  $z$  is the function of  $x$  and  $y$  &  $x$  and  $y$  are the functions of  $t$ .

$$z \rightarrow f(x,y) \rightarrow f(t)$$

$$\therefore z = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$$

**Direct differentiate w.r.t  $t$ ,**

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2} \times \frac{d}{dt}\left(\frac{2t}{1-t^2}\right) \\ &= \frac{2(1-t^2)^2}{(1-t^2)^2+4t^2} \times \left[ t \cdot \frac{1}{(1-t^2)^2} (-2t) + \frac{1}{1-t^2} \times 1 \right] \\ &= \frac{2(1-t^2)^2}{1+t^2} \times \frac{1}{(1-t^2)^2} \end{aligned}$$

$$\boxed{\therefore \frac{dz}{dt} = \frac{2}{1+t^2}}$$

**Hence Proved.**

**(e) Find the  $n$ th derivative of  $\cos 5x \cdot \cos 3x \cdot \cos x$ . [4]**

**Ans:** let  $y = \cos 5x \cdot \cos 3x \cdot \cos x$

$$= \frac{\cos(5x-3x) + \cos(5x+3x)}{2} \cdot \cos x$$

$$= \frac{1}{2} [\cos 2x \cdot \cos x + \cos 8x \cdot \cos x]$$

$$y = \frac{1}{4} [\cos 3x + \cos x + \cos 9x + \cos 7x]$$

**Take  $n$ th derivative,**

$$n \text{th derivative of } \cos(ax + b) = \frac{d^n}{dx^n} \cos\left(\frac{n\pi}{2} + ax + b\right)$$

$$y_n = \frac{1}{4} [9\cos\left(\frac{n\pi}{2} + 3x\right) + \cos\left(\frac{n\pi}{2} + x\right) + 81\cos\left(\frac{n\pi}{2} + 9x\right) + 49\cos\left(\frac{n\pi}{2} + 7x\right)]$$

**(f) Evaluate :  $\lim_{x \rightarrow 0} (x)^{\frac{1}{1-x}}$  [4]**

**Ans: Let**  $L = \lim_{x \rightarrow 0} (x)^{\frac{1}{1-x}}$

**Take log on both the sides,**

$$\therefore \log L = \lim_{x \rightarrow 0} \frac{\log x}{1-x}$$

**Apply L'Hospital rule ,**

$$\therefore \log L = \lim_{x \rightarrow 0} \frac{1}{x}$$

$$= 0$$

$$\therefore L = e^0 = 1$$

**Q.2(a) Find all values of  $(1 + i)^{1/3}$  & show that their continued**

**Product is  $(1+i)$ .**

**[6]**

**Ans: let**  $x = (1 + i)^{1/3}$

$$\therefore x^3 = 1 + i \sqrt{2} \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)$$

$$\therefore x^3 = \sqrt{2} \left[ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$$

**Add period  $2k\pi$  ,**

$$x^3 = \sqrt{2} \left[ \cos \left( \frac{\pi}{4} + 2k\pi \right) + i \sin \left( \frac{\pi}{4} + 2k\pi \right) \right]$$

**By applying De Moivre's theorem,**

$$x = \sqrt[3]{2} \left[ \cos \frac{1}{3} \left( \frac{\pi}{4} + 2k\pi \right) + i \sin \frac{1}{3} \left( \frac{\pi}{4} + 2k\pi \right) \right]$$

**where  $k=0,1,2$ .**

**Roots are :**

**Put  $k=0$**   $x_0 = \sqrt[3]{2} e^{i \frac{\pi}{12}}$

**Put  $k=1$**   $x_1 = \sqrt[3]{2} e^{i \frac{9\pi}{12}}$

**Put  $k=2$**   $x_2 = \sqrt[3]{2} e^{i \frac{17\pi}{12}}$

The continued product of roots is given by ,

$$\begin{aligned}
 x_0 x_1 x_2 &= \sqrt[2]{2} e^{i\frac{\pi}{12}} \times \sqrt[2]{2} e^{i\frac{9\pi}{12}} \times \sqrt[2]{2} e^{i\frac{17\pi}{12}} \\
 &= 16 \sqrt{2} e^{i\frac{27\pi}{12}} \\
 &= \sqrt{2} \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \\
 &= 1+i
 \end{aligned}$$

The continued product of roots is (1+i).

(b) Find non singular matrices P & Q such that PAQ is in normal form

Where  $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$  [6]

Ans : Matrix in PAQ form is given by ,

$$A = P A Q$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_3,$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 3R_1, R_3 - 2R_1,$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - 2C_1, C_3 + C_1,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{C_3}{5}$ ,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 1 \\ 0 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$C_2 + 6C_3$ ,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -4/5 & 1/5 \\ 0 & 1 & 0 \\ 0 & 6/5 & 1/5 \end{bmatrix}$$

$C_3 + C_2$ ,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -4/5 & -3/5 \\ 0 & 1 & 1 \\ 0 & 6/5 & 7/5 \end{bmatrix}$$

$-R_1$ ,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -4/5 & -3/5 \\ 0 & 1 & 1 \\ 0 & 6/5 & 7/5 \end{bmatrix}$$

Now A is in normal form with rank 3.

Compare with PAQ form,

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -4/5 & -3/5 \\ 0 & 1 & 1 \\ 0 & 6/5 & 7/5 \end{bmatrix}$$

(c) Find the maximum and minimum values of

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \quad [8]$$

Ans: given:  $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$f_x = 3x^2 + 3y^2 - 30x + 72 \quad f_{xx} = 6x - 30$$

$$f_y = 6xy - 30y \quad f_{yy} = 6x - 30$$

$$f_{xy} = 6y$$

**To find stationary values :**

$$f_x = 3x^2 + 3y^2 - 30x + 72 = 0 \quad \& \quad f_y = 6xy - 30y = 0$$

$$y=0 \text{ or } x=5$$

for  $y=0$  ,  $x=6,4$

$$\therefore (x,y)=(6,0), (4,0).$$

For  $x=5$  ,  $y=1,-1$

$$\therefore (x,y)=(5,1), (5,-1)$$

**Stationary points are : (6,0),(4,0),(5,1),(5,-1)**

**(i) For point (6,0) ,**

$$r = f_{xx} = 36 - 30 = 6, \quad s = f_{xy} = 0, \quad t = f_{yy} = 6$$

$$rt - s^2 = 36 > 0 \quad \text{and } r = 6 > 0$$

**function is minimum at (6,0).**

$$f_{min} = 108$$

**(ii) For point (4,0) ,**

$$r = f_{xx} = -6, \quad s = f_{xy} = 0, \quad t = f_{yy} = -6$$

$$rt - s^2 = 36 > 0 \quad \text{and } r = -6 < 0$$

**function is maximum at (4,0).**

$$f_{max} = 112$$

**(iii) for point (5,1) and (5,-1) ,**

**Thr points are neither maximum nor minimum.**

**$\therefore$  The maximum and minimum value of function are 112 and 108 .**

**Q.3(a)** If  $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ , show that  $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$  [6]

**Ans :** let  $u = f(r,s)$

$$\therefore r = \frac{y-x}{xy} \qquad \therefore s = \frac{z-x}{xz}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} \frac{1}{x^2} + \frac{\partial u}{\partial s} \left(\frac{-1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} = \frac{\partial u}{\partial r} \frac{(-1)}{y^2} + \frac{\partial u}{\partial s} (0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} = \frac{\partial u}{\partial r} (0) + \frac{\partial u}{\partial s} \left(\frac{1}{z^2}\right)$$

$$\therefore x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

Hence proved.

**(b)** Using encoding matrix  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , encode & decode the message

“MUMBAI”. [6]

**Ans :** Encoding matrix :  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

**Message is : MUMBAI.**

**The given message in matrix form is :**

$$B = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

**Encoded message in matrix form is given by ,**

$$C = AB$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$$



Encoded message is : 34 21 15 2 10 9

GUOJJI

Decoded matrix is given by ,

$$\begin{aligned} \mathbf{B} &= \mathbf{A}^{-1} \cdot \mathbf{C} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix} \\ \therefore \mathbf{B} &= \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix} \end{aligned}$$

Decoded message : MUMBAI

(c) Prove that  $\log\left[\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right)\right] = i \tan^{-1}(\sinh x)$  [8]

Ans : L.H.S =  $\log\left[\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right)\right]$

$$\begin{aligned} &= \log\left[\frac{1 + \tan\left(\frac{ix}{2}\right)}{1 - \tan\left(\frac{ix}{2}\right)}\right] \\ &= \log\left[1 + \tan\left(\frac{ix}{2}\right)\right] - \log\left[1 - \tan\left(\frac{ix}{2}\right)\right] \\ &= \log\left[1 + i \tanh\frac{x}{2}\right] - \log\left[1 - i \tanh\frac{x}{2}\right] \end{aligned}$$

We have ,

$$\log(a+ib) = \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1}\left(\frac{b}{a}\right)$$

$$\begin{aligned} \therefore &= \frac{1}{2} \log\left(1 + \tanh^2\frac{x}{2}\right) + i \tan^{-1}\left(\tanh\frac{x}{2}\right) - \frac{1}{2} \left[\log\left(1 + \tanh^2\frac{x}{2}\right) - i \tan^{-1}\left(\tanh\frac{x}{2}\right)\right] \\ &= 2i \left[\tan^{-1}\left(\tanh\frac{x}{2}\right)\right] \end{aligned}$$

$$\text{L.H.S} = 2i \cdot \tan^{-1}\left(\tanh\frac{x}{2}\right)$$

$$\text{R.H.S} = i \tan^{-1}(\sinh x)$$

We know that  $\sinh^{-1}x = \log(x + \sqrt{1+x^2})$

$$\tanh^{-1}x = \frac{1}{2} \left[\log\left(\frac{x+1}{1-x}\right)\right]$$

$$= i \tan^{-1} \left( \tanh \frac{x}{2} \right)$$

Also  $\sinh^{-1}(\tan x) = \tanh^{-1}(x)$

R.H.S =  $i \tan^{-1} \left( \tanh \frac{x}{2} \right)$

$$\log \left[ \tan \left( \frac{\pi}{4} + \frac{ix}{2} \right) \right] = i \tan^{-1}(\sinh x)$$

**Q.4(a) Obtain  $\tan 5\theta$  in terms of  $\tan \theta$  & show that**

$$1 - 10 \tan^2 \frac{x}{10} + 5 \tan^4 \frac{x}{10} = 0 \quad [6]$$

Ans: we have  $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Put  $n=5$ ,

$$\begin{aligned} \therefore \cos 5\theta + i \sin 5\theta &= (\cos \theta + i \sin \theta)^5 \\ &= \cos^5 \theta + 5 \cos^4 \theta \cdot i \sin \theta + 10 \cos^3 \theta \cdot (i \sin \theta)^2 \\ &\quad + 10 \cos^2 \theta \cdot (i \sin \theta)^3 + 5 \cos \theta (i \sin \theta)^4 + i \sin^5 \theta \\ &= [\cos^5 \theta - 10 \cos^3 \theta \cdot (\sin \theta)^2 \\ &\quad + 5 \cos \theta \cdot (\sin \theta)^4] + [5 \cos^4 \theta \cdot i \sin \theta \\ &\quad - 10 i \cos^2 \theta \cdot (\sin \theta)^3 + i \sin^5 \theta] \end{aligned}$$

**Compare real and imaginary parts**

$$\cos 5\theta = [\cos^5 \theta - 10 \cos^3 \theta \cdot (\sin \theta)^2 + 5 \cos \theta (\sin \theta)^4]$$

$$\sin 5\theta = +[5 \cos^4 \theta \cdot \sin \theta - 10 \cos^2 \theta \cdot (\sin \theta)^3 + \sin^5 \theta]$$

$$\tan 5\theta = \frac{[5 \cos^4 \theta \cdot \sin \theta - 10 \cos^2 \theta \cdot (\sin \theta)^3 + \sin^5 \theta]}{[\cos^5 \theta - 10 \cos^3 \theta \cdot (\sin \theta)^2 + 5 \cos \theta \cdot (\sin \theta)^4]}$$

$$\tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

put  $\theta = \frac{\pi}{10}$

$$1 - 10 \tan^2 \frac{x}{10} + 5 \tan^4 \frac{x}{10} = 0$$

(b) If  $y = e^{\tan^{-1} x}$ . Prove that

$$(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0 \quad [6]$$

Ans:  $y = e^{\tan^{-1} x} \dots\dots(1)$

Diff. w.r.t x,

$$y_1 = e^{\tan^{-1} x} \frac{1}{x^2+1}$$

$$(x^2 + 1)y_1 = e^{\tan^{-1} x} = y \quad \text{-----(from 1)}$$

Again diff. w.r.t x,

$$(x^2 + 1)y_2 + 2xy_1 = y \quad \dots\dots\dots(1)$$

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$$(uv)_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \dots + u v_n$$

$$u = (x^2 + 1), v = y \dots \text{for first term in eqn (1)}$$

$$u = 2x, v = y \dots \text{for second term in eqn (1)}$$

$$\therefore (1 + x^2)y_{n+2} + 2(n + 1)xy_{n+1} + n(n + 1)y_n - y_{n+1} = 0$$

$$\therefore (1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$$

Hence Proved.

(c) i. Express  $(2x^3 + 3x^2 - 8x + 7)$  in terms of  $(x-2)$  using Taylor's Series. [4]

ii. Prove that  $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

[4]

Ans: i. let  $f(x) = 2x^3 + 3x^2 - 8x + 7$

Here  $a = 2$

$$f(x) = 2x^3 + 3x^2 - 8x + 7 \quad f(2) = 19$$

$$f'(x) = 6x^2 + 6x - 8 \quad f'(2) = 28$$

$$f''(x) = 12x + 6 \quad f''(2) = 30$$

$$f'''(x) = f'''(2) = 12$$

Taylor's series is :

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

$$2x^3 + 3x^2 - 8x + 7 = 19 + 28(x - 2) + \frac{(x-2)^2}{2!}30 + \frac{(x-a)^3}{3!}12$$

$$2x^3 + 3x^2 - 8x + 7 = 19 + 28(x - 2) + 15(x - 2)^2 + 2(x - 2)^3$$

ii. let  $y = \tan^{-1} x$

diff. w.r.t x,

$$\therefore y_1 = \frac{1}{x^2+1}$$

Series expansion of  $y_1$ ,

We know that,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\therefore y_1 = 1 - x + x^2 - x^3 + \dots$$

Integrate  $y_1$  to find series expansion of y,

$$\therefore y = \int (1 - x + x^2 - x^3 + \dots) dx$$

$$\therefore y = x \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Hence Proved .

Q.5(a) If  $z = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$   $\partial$

Prove that  $\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$  [6]

Ans :  $z = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$

Diff. w.r.t. x partially ,

$$\begin{aligned} \frac{\partial z}{\partial x} &= x^2 \frac{x^2}{x^2 + y^2} \times \frac{-y}{x^2} + \tan^{-1} \frac{y}{x} \cdot 2x - 2y \frac{y^2}{x^2 + y^2} \times \frac{1}{y} \\ &= \frac{x^2}{x^2 + y^2} \times \frac{-y}{1} + 2x \tan^{-1} \frac{y}{x} - \frac{y^3}{x^2 + y^2} \end{aligned}$$

Diff. w.r.t y partially ,

$$\begin{aligned} \frac{\partial^2 z}{\partial y \partial x} &= -x^2 \left[ -y \cdot \frac{2y}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} \right] + 2 \frac{x^2}{x^2 + y^2} - \left[ -y^3 \cdot \frac{2y}{(x^2 + y^2)^2} + \frac{3y^2}{x^2 + y^2} \right] \\ &= \left[ \frac{2y^3 x^2}{(x^2 + y^2)^2} + \frac{-x^2}{x^2 + y^2} \right] + \frac{2x^2}{x^2 + y^2} + \frac{2y^4}{(x^2 + y^2)^2} - \frac{3y^2}{x^2 + y^2} \\ &= \frac{(x^2 - y^2)^2 \times (x^2 + y^2)^1}{(x^2 + y^2)^2 (x^2 - y^2)^1} \\ &= \frac{x^2 - y^2}{x^2 + y^2} \end{aligned}$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

Hence proved.

(b) Investigate for what values of  $\mu$  and  $\lambda$  the equations :  $2x + 3y + 5z = 9$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

Have (i) no solution (ii) unique solution (iii) Infinite value [6]

Ans : Given eqn :  $2x + 3y + 5z = 9$

$$7x+3y-2z=8$$

$$2x+3y+\lambda z=\mu$$

$$A X = B$$

$$\therefore \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$\text{Augmented matrix is : } \left[ \begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array} \right]$$

$$R_3 - R_1,$$

$$\rightarrow \left[ \begin{array}{ccc|c} 2 & 3 & 5 & 6 \\ 7 & 3 & -2 & 4 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array} \right]$$

(i) When  $\lambda=5, \mu \neq 9$  then  $r(A) = 2, r(A : B) = 3$

$$r(A) \neq r(A : B)$$

No Solution.

(ii) When  $\lambda \neq 5, \mu \neq 9, r(A) = r(A : B) = 3$

Unique solution exist.

(iii) When  $\lambda=5, \mu = 9, r(A) = r(A : B) = 2 < 3$

Infinite solution.

(c) Obtain the root of  $x^3 - x - 1 = 0$  by Newton Raphson Method

(upto three decimal places).

[8]

Ans : Equation :  $x^3 - 2x - 5 = 0$

$$\therefore f(x) = x^3 - 2x - 5$$

$$f(0) = -5 < 0 \text{ and } f(1) = -2 < 0 \text{ and } f(2) = 7 > 0$$

Root of given eqn lies between 1 and 2.

$$f'(x) = 3x^2 + 2$$

Let take  $x_0 = 2$

$$\begin{aligned}x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{7}{14} = 1.5\end{aligned}$$

Next iteration :

$$\begin{aligned}\therefore x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1.343 \\ \therefore x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.329\end{aligned}$$

For next iteration :

$$\begin{aligned}\therefore x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 1.329 \frac{f(1.329)}{f'(1.329)} \\ &= 1.3283\end{aligned}$$

The root of eqn is  $x = 1.3283$

Q.6(a) Find  $\tanh x$  if  $5\sinh x - \cosh x = 5$

[6]

Ans :  $5\sinh x - \cosh x = 5$

$$\text{But } \sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\therefore 5\left[\frac{e^x - e^{-x}}{2}\right] - \left[\frac{e^x + e^{-x}}{2}\right] = 5$$

$$\therefore 5e^x - 5e^{-x} - e^x - e^{-x} = 10$$

$$\therefore 4e^{2x} - 10e^x - 6 = 0$$

**Roots are :**  $e^x = 3, e = \frac{-1}{2}$

$$\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\left(\frac{-1}{2}\right) + 2}{-5/2} = \frac{-3}{5}$$

**Or**

$$\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{3 - 1/3}{3 + 1/3} = \frac{4}{5}$$

The values of  $\tanh x$  are :  $\frac{-3}{5}$  or  $\frac{4}{5}$

**(b)** If  $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ , Prove that i.  $xu_x + yu_y = \frac{1}{2} \tan u$

ii.  $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u \cos 2u}{4\cos^3 u}$  [6]

**Ans :**  $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put  $x = xt$  and  $y = yt$  to find degree.

$$\therefore u = \sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$$

$$\therefore \sin u = t^{1/2} \cdot \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{1}{t} \cdot f(x, y)$$

The function  $\sin u$  is homogeneous with degree  $\frac{1}{2}$ .

But  $\sin u$  is the function of  $u$  and  $u$  is the function of  $x$  and  $y$ .

By Euler's theorem ,

$$xu_x + yu_y = Q(u) = n \frac{f(u)}{f'(u)} = \frac{1}{2} \tan u$$

$$\therefore xu_x + yu_y = \frac{1}{2} \tan u$$

$$\begin{aligned} \therefore x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} &= Q(u)[G'(u) - 1] \\ &= \frac{1}{2} \tan u \left[ \frac{\sec^2 u - 2}{2} \right] \end{aligned}$$



$$= \frac{1}{4} \tan u \left[ \frac{\tan^2 u - 1}{1} \right]$$

$$= \frac{1}{4} \times \frac{\sin u}{\cos u} \left[ \frac{\sin^2 u - \cos^2 u}{\cos^2 u} \right]$$

$$\therefore x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \frac{-\sin u \cos 2u}{4 \cos^3 u}$$

Hence Proved.

(c) Solve the following system of equation by Gauss Siedal Method,

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

[8]

**Ans:** By Gauss Seidal method ,

Given eqn :  $20x + y - 2z = 17$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

From given eqn :  $|20| > |1| + |-2|$

$$|20| > |3| + |-1|$$

$$|20| > |2| + |-3|$$

The given eqn are in correct order.

$$\therefore x = \frac{1}{20} [17 - y + 2z]$$

$$\therefore y = \frac{1}{20} [-18 - 3x + z]$$

$$\therefore z = \frac{1}{20} [25 - 2x + 3y]$$

i) For 1<sup>st</sup> iteration : take  $y = 0, z = 0$

$$x = \frac{1}{20} [17] = 0.85$$

$$x = 0.85, z = 0 \text{ gives } y = -1.0275$$

$$x = 0.85, y = -1.0275 \text{ gives } x_3 = 1.0109$$

II) For 2<sup>nd</sup> iteration : take  $y = -1.0275, z = 1.0109$

$$x = \frac{1}{20} [17 + 1.0275 - 2(1.0109)1.0025]$$

$$x = 1.0025, z = 1.0109 \text{ gives } y = -0.9998$$

$$x = 1.0025, y = -0.9998 \text{ gives } z = 0.9998$$

III) For 3<sup>rd</sup> iteration :  $y = -0.9998, z = 0.9998$

$$x_1 = \frac{1}{20} [17 + 0.9998 + 2(0.9998)1.00]$$

$$x = 1.00, z = 0.9998 \text{ gives } y = -1.00$$

$$x = 1.00, y = -1.00 \text{ gives } z = 1.00$$

**Result :  $x = 1.00, y = -1.00, z = 1.00$**